COUNTING SPANNING TREES ON FRACTAL GRAPHS

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Jason Allen Anema

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Jason Allen Anema, Ph.D.

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Using the method of spectral decimation and a modified version of Kirchhoff's Matrix-Tree Theorem, a closed form solution to the number of spanning trees on approximating graphs to a fully symmetric self-similar structure on a finitely ramified fractal is given. Examples calculated include the Sierpiński Gasket, a non-p.c.f. analog of the Sierpiński Gasket, the Diamond fractal, and the Hexagasket. For each example, the asymptotic complexity constant is found.

Dropping the fully symmetry assumption, it is shown that the limsup and liminf of the asymptotic complexity constant exist. Calculating the number of spanning trees on the m-Tree fractal shows that the asymptotic complexity constant for this class of fractals has no upper bound.

BIOGRAPHICAL SKETCH

Jason Anema was born in Indianapolis Indiana, USA on the 30th of June, 1983. His family consist of his parents Joyce M. and Gregory F., younger brothers Aaron N. and Kevin B., grandparents Burney J. Scott and Bettie G. Scott, aunt Tina M., and nephew Gage A. He graduated from Purdue University with a B.S. in Mathematics, Statistics, and Actuarial Science, each with honors and distinction, and minors in Economics and Management. He began his graduate work at Cornell University in 2005, received his M.S. in Mathematics in 2009, and completed his dissertation under the supervision of Professor Robert Strichartz.

This document is dedicated to all of my loving family and friends.

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CHAPTER 1

INTRODUCTION AND DEFINITIONS

1.1 Introduction

The Laplacian on fractals, as a counterpart to Laplacians on smooth Riemannian manifolds, have been intensively studied. There is a vast amount of mathematics and physics literature devoted to analysis on fractals. The Laplacian on the Sierpiński Gasket was introduced in the physics literature [2, 46, 47], where the spectral decimation method was developed, and was first constructed as the generator of a diffusion process by S. Goldstein and S. Kusuoka in [39, 30]. This method of construction is known as the probabilistic approach. The following year, M. Barlow and E. Perkins [6] presented a detailed study of the properties of this diffusion process, obtaining an Aronson-type estimate of the heat kernel on the Sierpiński Gasket. In [41] Lindstrøm extended the construction of this diffusion process to nested fractals. J. Kigami and S. Kusuoka developed an analytic approach to constructing the Laplacian using the theory of Dirichlet forms in [35, 36, 40]. This approach to the theory of the Laplacian was developed for post-critically finite (finitely ramified) self similar sets and nested fractals, and is summarized in Kigami's book, which has an extensive reference list [37]. Some advantages of this approach are that one can describe harmonic functions, Green's function and solution's to Poisson's equations. Many nice features of analysis on fractals have been discovered by R. Strichartz, A. Teplyaev, and others, in [34, 48, 52, 53, 3, 4, 54, 55, 56, and references therein]. In [3, 4] A. Teplyaev, B. Steinhurst, et al., describe the method of spectral decimation for self-similar fully symmetric finitely ramified fractals, which shows how to explicitly calculate the spectrum of the Laplacian on such fractals, generalizing the ideas of [26, 49]. The central result of the present work relies on their paper to describe how to calculate, in an analytic fashion, the number of spanning trees of the sequence of graph approximations to such fractals.

The problem of counting the number of spanning trees in a finite graph dates back more than 150 years. It is one of the oldest and most important graph invariants, and has been actively studied for decades. Kirchhoff's famous Matrix-Tree Theorem [38], appearing in 1847, relates properties of electrical networks and the number spanning trees. There are now a large variety of proofs for the Matrix-Tree Theorem, for some examples see [10, 15, 32]. Counting spanning trees is a problem of fundamental interest in mathematics [9, 64, 14, 42, 11, e.g.] and physics [65, 67, 25, 66, 22, e.g.]. Its relation to probability theory was explored in [43, 45]. It has found applications in theoretical chemistry, relating to the enumeration of certain chemical isomers [13], and as a measure of network reliability in the theory of networks [19].

Recently, S.C. Chang et al. studied the number of spanning trees and the associated asymptotic complexity constants on regular lattices in [17, 18, 51, 63]. These types of problems led them to consider spanning trees on self-similar fractal lattices, as they exhibit scale invariance rather than translation invariance. In [16] S.C. Chang, L.C. Chen, and W.S. Yang calculate the number of spanning trees on the sequence of graph approximations to the Sierpiński Gasket of dimension two, three and four, as well as for two generalized Sierpiński Gaskets ($SG_{2,3}(n)$ and $SG_{2,4}(n)$), and conjecture a formula for the number of spanning trees on the $d-dimensional$ Sierpiński Gasket at stage n, for general d. Their method of proof uses a decomposition argument to derive multi-dimensional polynomial recursion equations to be solved. Independently, that same year, E. Teufl and S. Wagner [57] give the number of spanning trees on the Sierpiński Gasket of dimension two at stage n, using the same argument. In [58] they expand on this work, contructing graphs by a replacement procedure yielding a sequence of self-similar graphs (this notion of self-similarity is different than in [37]), which include the Sierpiński graphs. For a variety of enumeration problems, including counting spanning trees, they show that their construction leads to polynomial systems of recurrences and provide methods to solve these recurrences asymptotically. Using the same construction technique in [59], they give, under the assumptions of $strong\ symmetry$ (see [59, section 2.2]) and $connectedness$, a closed form equation for the number of spanning trees [59, Theorem 4.2]. This formulation requires calculating the resistance scaling factor and the tree scaling factor (defined in [59, Theorem 4.1]). In Section 8.3.1 they show that the $d-dimensional$ Sierpiński Gasket at stage n, satisfies their assumptions and prove the conjecture of [16].

$Strong\ Symmetry$ is a condition which must be satisfied on each level of construction, whereas the $full\ symmetry$ condition, that will be assumed in the present work, is only a condition on the first level of construction. Sequences of graphs ,in this work, will also be self-similar (in the sense of J. Kigami [37]), and finitely ramified. Under these assumptions, Theorem 2.3.5 gives a closed formula for the number of spanning trees on the approximating graphs to a fully symmetric self-similar structure on a finitely ramified fractal. This formula requires one to carry out spectral decimation as in [3], see the proof of Theorem 2.3.5 for details. The beginning of Chapter 2 is dedicated to building up some auxiliary results including Lemma 2.2.1, which relates the coefficients of the characterstic polynomial of the Graph Laplacian and the Probabilistic Graph

Laplacian, and Kirchhoff's Matrix-Tree Theorem for Probabilistic Graph Laplacians (Theorem 2.3.1). These are essential to this work since spectral decimation only works for the Probabilistic Graph Laplacian, or a multiple of it, as noted in [3, Remark 3.3]. In Theorem 2.4.2, the assumption of full symmetry is dropped, and the existence of the limsup and liminf of the asymptotic complexity constant is shown. Theorem 2.4.4 shows that this constant can be arbitrarily large within this class of fractal graphs using the m-Tree Fractals of Section 3.6.

Chapter 3 is dedicated to calculating the number of spanning trees for on specific fractals. The Sierpiński graphs are examples of graphs which are both strongly symmetric and fully symmetric. In Section 3.1 an alternate proof of the number of spanning trees on $SG_2(n)$ is given to illustrate how to use Theorem 2.3.5. The Hexagasket is an example of a fully symmetric self-similar structure on a finitely ramified fractal which is not strongly symmetric. The number of spanning trees on the graph approximations to the Hexagasket are calculated in Section 3.4 using Theorem 2.3.5. Other examples worked are a non-p.c.f. analog of the Sierpiński Gasket in Section 3.2, the Diamond Fractal in Section 3.3, $SG_{2,3}(n)$ (providing an alternate proof of [16, Theorem 4.1]) in Section 3.5, and the m-Tree Fractal in Section 3.6.

1.2 Definitions

Definition 1.2.1. For any graph $T = (V, E)$ having n labelled vertices $v_1, v_2, ..., v_n$, the **adjacency matrix A** on T is defined by

$$A = \big((a_{ij})\big)$$

where a_{ij} is the number of copies of $\{v_i, v_j\} \in E$

During the course of this work, all graphs are assumed to be loopless, meaning that $\{v_i, v_i\} \notin E$ for any $1 \leq i \leq n$. In the setting of this text, this is a natural assumption, as all fractal graphs are loopless.

Definition 1.2.2. For any graph $T = (V, E)$ having n labelled vertices $v_1, v_2, ..., v_n$, the **degree matrix D** on T is defined by

$$D = \big((d_{ij})\big)$$

where $d_{ij} = 0$ for $i \neq j$, and $d_{ii} = \deg(v_i)$ which is the number of non-loop edges containing v_i plus twice the number of loops containing v_i

Definition 1.2.3. For any graph $T = (V, E)$ having n labelled vertices $v_1, v_2, ..., v_n$, the **graph Laplacian G** on T is defined by

$$G = D - A$$

where D is the degree matrix on T, and A is the adjacency matrix on T.

Definition 1.2.4. For any graph $T = (V, E)$ having n labelled vertices $v_1, v_2, ..., v_n$, where none of the vertices are isolated, the **probabilistic graph Laplacian P** on T is defined by

$$P = D^{-1}G$$

where D^{-1} is the inverse of the degree matrix on T, and G is the graph Laplacian on T.

Definition 1.2.5. Let $T = (V_T, E_T)$ be a graph, and $S = (V_S, E_S)$ be any subgraph of T. If $V_S = V_T$ and S is a $tree$, then S is a **spanning tree** of T.

Definition 1.2.6. Let T_n for $n \geq 0$ be a sequence of finite graphs, $|T_n|$ the number of vertices in T_n, and $\tau(T_n)$ denote the number of spanning trees of T_n. $\tau(T_n)$ is called the **complexity** of T_n. The **asymptotic complexity** of the sequence T_n is defined as

$$\lim_{n\to\infty} \frac{log(\tau(T_n))}{|T_n|}.$$

When this limit exist, it is called the **asymptotic complexity constant**, or the **tree entropy** of T_n, or the **thermodynamic limit** of T_n.

Definition 1.2.7. As in [37], let (X, d) be a complete metric space. If $f_i : X \to X$ is a contraction with respect to the metric d for $i = 1, 2, ...\ m$, then there exist a unique non-empty compact subset K of X that satisfies

$$K = f_1(K) \cup \cdots \cup f_m(K).$$

K is called the **self-similar set with respect to** $\{\mathbf{f_1}, \mathbf{f_2}, ...\mathbf{f_m}\}$

Definition 1.2.8. As in [3], if K is a self-similar set with respect to $\{f_1, f_2, ...f_m\}$ such that each f_i is injective and for any n and for any two distinct words ω, ω'

$\in W_n$=$\{1, ...m\}^n$ we have

$$K_\omega \cap K_{\omega'} = F_\omega \cap F_{\omega'}$$

where f_ω=$f_{\omega_1} \circ \cdot \cdot \cdot \circ f_{\omega_n}$, K_ω=$f_\omega(K)$, F_0 is the set of fixed points of $\{f_1, f_2, ...f_m\}$, and $F_\omega = f_\omega(F_0)$, is called a **finitely ramified self-similar set with respect to $\{\mathbf{f_1}, \mathbf{f_2}, ...\mathbf{f_m}\}$**

Definition 1.2.9. Let K be a self-similar set with respect to $\{f_1, f_2, ...f_m\}$. There is a natural sequence of **approximating graphs $\mathbf{V_n}$** with vertex set F_n defined as follows. For all $n \geq 0$ and for all $\omega \in W_n$ define V_0 as the complete graph with vertices F_0,

$$F_n := \bigcup_{\omega \in W_n} F_\omega,$$
$$F_\omega := \bigcup_{x \in V_)} F_\omega(x),$$

where $F_\omega := f_{a_n} \circ f_{a_{n-1}} \circ \cdots f_{a_1} \circ$ and $\omega = a_1 a_2 \cdots a_n$. Also, $x, y \in F_n$ are connected by an edge in V_n if $f_i^{-1}(x)$ and $f_i^{-1}(y)$ are connected by an edge in V_{n-1} for some $1 \leq i \leq m$.

Definition 1.2.10. As in [37], let K be a compact metrizable topological space and S be a finite set. Also, let F_i be a continuous injection from K to itself $\forall i \in S$. Then, $(K, S, \{F_i\}_{i \in S})$ is called a **self-similar structure** if there exists a continuous surjection $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i \ \forall i \in S$, where $\Sigma = S^{\mathbb{N}}$ the one-sided infinite sequences of symbols in S and $\sigma_i : \Sigma \to \Sigma$ is defined by $\sigma_i(\omega_1\omega_2\omega_3...) = i\omega_1\omega_2\omega_3...$ for each $\omega_1\omega_2\omega_3... \in \Sigma$

Clearly if K is the self-similar set with respect to injective contractions $\{f_1, f_2, ...f_m\}$, then $(K, \{1, 2, ...m\}, \{f_i\}_{i=1}^m)$ is a self-similar structure.

Definition 1.2.11. As in [37], let $L_j = (K_j, S_j, \{F_i^{(j)}\}_{i \in S_j})$ be self-similar structures and $\Sigma(S_j)$ be the one-sided infinite sequences of symbols in S_j for $j = 1, 2$.

Also let $\pi_j : \Sigma(S_j) \to K_j$ be the continuous surjection association with L_i for $j = 1, 2$. We say that $\mathbf{L_1}$ **and** $\mathbf{L_2}$ **are isomorphic** if there exist a bijective map $\rho : S_1 \to S_2$ such that $\pi_2 \circ \iota_\rho \circ \pi_1^{-1}$ is a well-defined homeomorphism between K_2 and K_1, where ι_ρ is the natural bijective map induced by ρ, i.e.$\iota_\rho(\omega_1\omega_2...) = \rho(\omega_1)\rho(\omega_2)...$ We say that two self-similar structures are the same if they are isomorphic.

Notice that two non-isomorphic self-similar structures can have the same finitely ramified self-similar set, however the structures will not have the same sequence of approximating graphs V_n. Also, any two isomorphic self-similar structures whose compact metrizable topological spaces are finitely ramified self-similar sets will have approximating graphs which are isomorphic $\forall n \geq 0$.

Definition 1.2.12. A **fully symmetric** finitely ramified self-similar structure with respect to $\{f_1, f_2, ...f_m\}$ is a self-similar structure $(K, \{1, 2, ...m\}, \{f_1, f_2, ...f_m\})$ such that K is a finitely ramified self-similar set, and, as in [3], for any permutation $\sigma : F_0 \to F_0$ there is an isometry $g_\sigma : K \to K$ that maps any $x \in F_0$ into $\sigma(x)$ and preserves the self-similar structure of K. This means that there is a map $\tilde{g_\sigma} : W_1 \to W_1$ such that $f_i \circ g_\sigma = g_\sigma \circ f_{\tilde{g_\sigma}(i)}$ $\forall i \in W_1$. The group of isometries g_σ is denoted $\mathfrak{G}$.

As in [33], the definition of a fully symmetric finitely ramified self-similar structure may be combined into one compact definition.

Definition 1.2.13. A fractal K is a fully symmetric finitely ramified self-similar set if K is a compact connected metric space with injective contraction maps on a complete metric space $\{f_i\}_{i=1}^m$ such that

$$K = f_1(K) \cup \cdots \cup f_m(K).$$

and the following three conditions hold:

1. there exist a finite subset F_0 of K such that

$$f_j(K) \cap f_k(K) = f_j(F_0) \cap f_k(F_0)$$

   for $j \neq k$ (this intersection may be empty);

2. if $v_0 \in F_0 \cap f_j(K)$ then v_0 is the fixed point of f_j;

3. there is a group $\mathcal{G}$ of isometries of K that has a doubly transitive action on F_0 and is compatible with the self-similar structure $\{f_i\}_{i=1}^m$, which means that for any j and any $g \in \mathcal{G}$ there exist a k such that

$$g^{-1} \circ f_j \circ g = f_k.$$

CHAPTER 2

GRAPH LAPLACIANS AND KIRCHHOFF'S MATRIX-TREE THEOREM

2.1 Matrix Decompositions of Graph Laplacians

Fix a graph T having n labelled vertices $v_1, v_2, ..., v_n$. Let G be its graph Laplacian and P be its probabilistic graph Laplacian, then $G = DP$.

Let I be the $n \times n$ identity matrix,

$$\chi(G) = |G - xI| = \sum_{i=0}^{n} c_i^G x^i$$

be the characteristic polynomial of G, and

$$\chi(P) = |P - xI| = \sum_{i=0}^{n} c_i^P x^i$$

be the characteristic polynomial of P.

Let $S := \{1, 2, ..., n-1, n\}$. If $\theta \subseteq S$, then let $\bar{\theta}$ denote the complement of θ in S. For any $n \times n$ matrix A and any $\theta \subseteq S$, let $A(\theta)$ denote the principal submatrix of A formed by deleting all rows and columns not indexed by an element of θ.

Example 2.1.1. Let $n = 4$, $A = \big((a_{ij})\big)$, and $\theta = \{1, 3\}$. Then

$$A(\theta) = \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} \qquad \text{and} \qquad A(\bar{\theta}) = \begin{pmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{pmatrix}.$$

By convention, if $\theta = \emptyset$, then $A(\theta)$ is taken to be the identity matrix of order one.

Notice that for any $n \times n$ diagonal matrix A and any $n \times n$ matrix B, we have

$$[AB]\,(\theta) = [A(\theta)]\,[B(\theta)]\,.$$

Proposition 2.1.2 (Collings,[20])**.** Let D be an $m \times m$ diagonal matrix and let A be an arbitrary $m \times m$ matrix. The determinant of $(D + A)$ is given by

$$|D + A| = \sum_{\theta \subseteq S} |D(\bar{\theta})| \cdot |A(\theta)|,$$

where the summation is over all subsets $S = \{1, ..., m\}$.

2.2 Relating Characteristic Polynomials Between Graph and Probabilistic Graph Laplacians

2.2.1 Coefficients of Characteristic Polynomials of G and P

Lemma 2.2.1. For any graph T with n vertices, the coefficient of $\chi(G)$ and $\chi(P)$ are given by

$$c^G_{n-i} = (-1)^{n-i} \sum_{|\theta|=i} |D(\theta)| \cdot |P(\theta)| \tag{2.1}$$

and

$$c^P_{n-i} = (-1)^{n-i} \sum_{|\theta|=i} |P(\theta)|. \tag{2.2}$$

Proof. We have by Proposition 2.1.2 above and term expansion, that

$$
\begin{aligned}
\chi(P) = |(-xI)+P| &= \sum_{\theta\subseteq S} |-xI(\bar{\theta})|\cdot|P(\theta)| \\
&= \sum_{i=0}^{n}\sum_{|\theta|=i} |P(\theta)|\cdot|-xI(\bar{\theta})| \\
&= \sum_{i=0}^{n}(-x)^{n-i}\sum_{|\theta|=i}|P(\theta)|
\end{aligned}
$$

Similarly, we have

$$
\chi(G) = |(-xI)+G| = \sum_{i=0}^{n}(-x)^{n-i}\sum_{|\theta|=i}|G(\theta)|
$$

Now using $G = DP$ and $G(\theta) = D(\theta)P(\theta)$, we have

$$
\begin{aligned}
\chi(G) = |(-xI)+G| &= \sum_{i=0}^{n}(-x)^{n-i}\sum_{|\theta|=i}|G(\theta)| \\
&= \sum_{i=0}^{n}(-x)^{n-i}\sum_{|\theta|=i}|D(\theta)P(\theta)| \\
&= \sum_{i=0}^{n}(-x)^{n-i}\sum_{|\theta|=i}|D(\theta)|\cdot|P(\theta)|,
\end{aligned}
$$

where the last line follows from $\det(AB) = \det(A)\det(B)$. Examination of the coefficients immediately gives us

$$
c^{G}_{n-i} = (-1)^{n-i}\sum_{|\theta|=i}|D(\theta)|\cdot|P(\theta)|
$$

and

$$
c^{P}_{n-i} = (-1)^{n-i}\sum_{|\theta|=i}|P(\theta)|,
$$

as desired. □

We now quote the version of Kirchhoff's Matrix-Tree Theorem which will be used in this work.

Proposition 2.2.2. (Kirchhoff's Matrix-Tree Theorem for Graph Laplacians, [38, 60]) For any connected loopless graph T with n labelled vertices, the number of spanning trees of T is

$$\tau(T) = |det(G')| = \frac{1}{n}|\prod_{j=1}^{n-1} \lambda_j^G|,$$

where G' is any cofactor of T's Graph Laplacian G and $\lambda_1^G, ..., \lambda_{n-1}^G$ are the non-zero eigenvalues of G.

Theorem 2.2.3. For any connected graph T with n vertices $\{v_1, ..., v_n\}$, we have that

$$c_1^G = n \cdot (-1)^{1-n} \frac{\left(\prod_{j=1}^{n} \deg(v_j)\right)}{\left(\sum_{j=1}^{n} \deg(v_j)\right)} \cdot c_1^P.$$

Proof. Let $\lambda_1^G, ..., \lambda_{n-1}^G$ be the non-zero eigenvalues of G and let $\lambda_1^P, ..., \lambda_{n-1}^P$ be the non-zero eigenvalues of P. Let $\theta_i = S \backslash \{i\}$. From Kirchhoff's Matrix-Tree Theorem (Proposition 2.2.2) we know that $\forall i \in S$

$$|G(\theta_i)| = \pm \frac{1}{n} \prod_{j=1}^{n-1} \lambda_j^G,$$

and it is easy to see that $\forall i \in S$, $|G(\theta_i)|$ has the same sign. Combining this with Equation 2.2.1 in Lemma 2.2.1, we see that

$$c_{n-i}^G = (-1)^{n-i} \sum_{|\theta|=i} |G(\theta)| = (-1)^{n-1} \prod_{j=1}^{n-1} \lambda_j^G.$$

This followed from $\chi(G) = x \cdot \prod_{j=1}^{n-1} \left(x - \lambda_j^G\right)$, as G has only one zero eigenvalue (since T is connected.)

Hence, $\forall i \in S$,

$$|G(\theta_i)| = \frac{(-1)^{n-1}}{n} \prod_{j=1}^{n-1} \lambda_j^G.$$

Now from the previous lemma and the same observations as above,

$$c_1^P = (-1)^{n-1} \sum_{|\theta|=n-1} |P(\theta)| = (-1)^{n-1} \prod_{j=1}^{n-1} \lambda_j^P.$$

Let $d_j := \deg(v_j)$. Then $\forall i \in S$,

$$\begin{aligned} |G(\theta_i)| &= |D(\theta_i)| \cdot |P(\theta_i)| \\ &= \left(\prod_{\substack{j \neq i \\ j \in S}} d_j \right) |P(\theta_i)|. \end{aligned}$$

Also $\forall i \in S$, we have

$$|G(\theta_i)| = \frac{(-1)^{n-1}}{n} \prod_{j=1}^{n-1} \lambda_j^G.$$

Combining these, we have

$$\frac{\left(\frac{(-1)^{n-1}}{n} \prod_{j=1}^{n-1} \lambda_j^G \right)}{\left(\prod_{j \neq i} d_j \right)} = |P(\theta_i)|. \tag{2.3}$$

Taking Equation 2.3 and summing over $i = 1, ..., n$ we have

$$\sum_{i=1}^{n} \left(\frac{1}{\prod_{j \neq i} d_j} \right) \cdot \left(\frac{(-1)^{n-1}}{n} \prod_{j=1}^{n-1} \lambda_j^G \right) = \sum_{i=1}^{n} |P(\theta_i)| = \sum_{|\theta|=n-1} |P(\theta)|.$$

The left-hand side of this equality becomes

$$\frac{\left(\frac{(-1)^{n-1}}{n} \right) \left(\sum_{i=1}^{n} d_i \right)}{\left(\prod_{i=1}^{n} d_i \right)} \prod_{j=1}^{n-1} \lambda_j^G,$$

while the right-hand side is

$$(-1)^{1-n} c_1^P.$$

Hence, we see that

$$\frac{(-1)^{1-n}}{n} \prod_{j=1}^{n-1} \lambda_j^G = \frac{\left(\sum_{i=1}^{n} d_i\right)}{\left(\prod_{i=1}^{n} d_i\right)} (-1)^{1-n} \cdot c_1^P.$$

Since $c_1^G = (-1)^{n-1} \prod_{j=1}^{n-1} \lambda_j^G$, we have

$$c_1^G = n \cdot (-1)^{1-n} \frac{\left(\prod_{j=1}^{n} \deg(v_j)\right)}{\left(\sum_{j=1}^{n} \deg(v_j)\right)} \cdot c_1^P$$

□

2.3 Proof of Main Theorem

Theorem 2.3.1 (Kirchhoff's Matrix-Tree Theorem for Probabilistic Graph Laplacians)**.** For any connected graph T with n labelled vertices, the number of spanning trees of T is

$$\tau(T) = \left| \frac{\left(\prod_{j=1}^{n} d_j\right)}{\left(\sum_{j=1}^{n} d_j\right)} \left(\prod_{j=1}^{n-1} \lambda_j^P\right) \right|.$$

Proof of Theorem 2.3.1. From Kirchhoff's Matrix-Tree Theorem (Proposition 2.2.2 we know that

$$\tau(T) = \left| \frac{1}{n} c_1^G \right|.$$

From Theorem 2.2.3, we have

$$c_1^G = n \cdot (-1)^{1-n} \frac{\left(\prod_{j=1}^{n} \deg(v_j)\right)}{\left(\sum_{j=1}^{n} \deg(v_j)\right)} \cdot c_1^P.$$

Also, we know $c_1^P = (-1)^{n-1} \prod_{j=1}^{n-1} \lambda_j^P$. So we have that

$$\tau(T) = \left|\frac{1}{n} c_1^G\right| = \left| \frac{\left(\prod_{j=1}^{n} d_j\right)}{\left(\sum_{j=1}^{n} d_j\right)} \left(\prod_{j=1}^{n-1} \lambda_j^P\right) \right|.$$

□

For the remainder of this section let K be a fully symmetric finitely ramified self-similar structure, V_n be its sequence of approximating graphs, and P_n denote the probabilistic graph Laplacian of V_n.

The next two Propositions describe the spectral decimation process, which inductively gives the spectrum of P_n.

The V_0 network is the complete graph on the boundary set and we set $m = |V_0|$.

Write P_1 in block form

$$P_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is a square block matrix associated to the boundary points. Since the V_1 network never has an edge joining two boundary points A is the mxm identity matrix. The Schur Complement of P_1 is

$$S(z) = (A - zI) - B(D - z)^{-1} C$$

Proposition 2.3.2. (Bajorin, et al.,[3]) For a given fully symmetric finitely ramified self-similar structure K there are unique scalar valued rational functions

$\phi(z)$ and $R(z)$ such that for $z \notin \sigma(D)$

$$S(z) = \phi(z)(P_0 - R(z))$$

Now P_0 has entries $a_{ii} = 1$ and $a_{ij} = \frac{-1}{m-1}$ for $i \neq j$. Looking at specific entries of this matrix valued equation we get two scalar valued equations

$$\phi(z) = -(m-1)S_{1,2}(z)$$

and

$$R(z) = 1 - \frac{S_{1,1}}{\phi(z)}.$$

Where $S_{i,j}$ is the i, j entry of the matrix $S(z)$.

Now, we let

$$E(P_0, P_1) := \sigma(D) \bigcup \{z : \phi(z) = 0\}$$

and call $E(P_0, P_1)$ the exceptional set.

Let $mult_D(z)$ be the multiplicity of z as an eigenvalue of D, $mult_n(z)$ be the multiplicity of z as an eigenvalue of P_n, $mult_n(z) = 0$ if and only if z is not an eigenvalue of P_n, and similarly $mult_D(z) = 0$ if and only if z is not and eigenvalue of D. Then we may inductively find the spectrum of P_n with the following Proposition.

Proposition 2.3.3. (Bajorin, et al.,[3]) For a given fully symmetric finitely ramified self-similar structure K, and $R(z),\ \phi(z),\ E(P_0, P_1)$ as above, the spectrum of P_n may be calculate inductively using the following criteria:

1. if $z \notin E(P_0, P_1)$, then

$$mult_n(z) = mult_{n-1}(R(z))$$

2. if $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a removable singularity at z then,

$$mult_n(z) = |V_{n-1}|$$

3. if $z \in \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at z, $R(z)$ has a removable singularity at z, and $\frac{\partial}{\partial z}R(z) \neq 0$, then

$$mult_n(z) = m^{n-1}mult_D(z) - |V_{n-1}| + mult_{n-1}(R(z))$$

4. if $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z, and $\phi(z) \neq 0$,then

$$mult_n(z) = m^{n-1}mult_D(z) + mult_{n-1}(R(z))$$

5. if $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z, and $\phi(z) = 0$,then

$$mult_n(z) = m^{n-1}mult_D(z) + |V_{n-1}| + mult_{n-1}(R(z))$$

6. if $z \in \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at z, $R(z)$ has a removable singularity at z, and $\frac{\partial}{\partial z}R(z) = 0$, then

$$mult_n(z) = m^{n-1}mult_D(z) - |V_{n-1}| + 2mult_{n-1}(R(z))$$

7. if $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a pole at z, then $mult_n(z) = 0$.
8. if $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z, $\phi(z) = 0$ and $R(z)$ has a pole at z, then

$$mult_n(z) = m^{n-1}mult_D(z).$$

After carrying out the inductive calculations using items (1)-(8), define

$$A := \{\alpha : \alpha \text{ satisfies item (2) or (8)}\}$$

for $\alpha \in A$, $\alpha_n := mult_n(\alpha)$

$$B := \{\beta : \text{ for some } n \geq 1,\ mult_n(\beta) \neq 0 \text{ and } mult_{n-1}(R(\beta)) \neq 0\}$$

and for $\beta \in B$, $\beta_n^k := mult_n(R_{(-k)}(\beta))$.

Since V_n is connected $mult_n(0) = 1$ for all $n \geq 0$. Again from [3], we get that

$$\sigma(P_n) \setminus \{0\} = \bigcup_{\alpha \in A} \{\alpha\} \bigcup_{\beta \in B} \left[\bigcup_{k=0}^{n} \{R_{-k}(\beta) : \beta_n^k \neq 0\} \right].$$

Theorem 2.3.4. Let $R(z)$ be a rational function such that $R(0) = 0$, $\deg(R(z)) = d$, $R(z) = \frac{P(z)}{Q(z)}$, with $\deg(P(z)) > \deg(Q(z))$. Let P_d be the leading coefficient of $P(z)$. Fix $\alpha \in \mathbb{C}$. Let $\{R_{(-n)}(\alpha)\}$ be the set of n^{th} preiterates of α under $R(z)$. By convention, $R_{(0)}(\alpha) := \{\alpha\}$. Then for $n \geq 0$,

$$\prod_{z \in \{R_{(-n)}(\alpha)\}} z = \alpha \left(\frac{-Q(0)}{P_d} \right)^{\left(\frac{d^n - 1}{d-1}\right)}.$$

Proof of Theorem 2.3.4. We prove by induction.

For $n = 0$, the result is clear. For $n = 1$, we note

$$\begin{aligned} \{R_{(-1)}(\alpha)\} &= \{z : R(z) = \alpha\} \\ &= \{z : P(z) - \alpha Q(z) = 0\} \\ &= \{z : P_d z^d + \cdots - Q(0)\alpha = 0\}, \end{aligned}$$

where $Q(0)$ is the constant term of $Q(z)$. As the product of the roots of a polynomial is equal to the constant term over the coefficient of the highest degree term, we have that

$$\prod_{z \in \{R_{(-1)}(\alpha)\}} z = \frac{-\alpha Q(0)}{P_d}.$$

Assume our equation holds for n. Then for $n+1$ we have

$$\big\{w : w \in R_{(-(n+1))}(\alpha)\big\} = \big\{R_{(-1)}(w) : w \in R_{(-n)}(\alpha)\big\}.$$

So

$$\begin{aligned}
\prod_{w \in \left\{R_{(-(n+1))}(\alpha)\right\}} w &= \prod_{w \in \left\{R_{(-n)}(\alpha)\right\}} \left(\prod_{z \in \left\{R_{(-1)}(w)\right\}} z \right). \\
&= \prod_{w \in \left\{R_{(-n)}(\alpha)\right\}} \left(\frac{-wQ(0)}{P_d} \right),
\end{aligned}$$

with the second equality following from the $n = 1$ case.

Since $\left|R_{(-n)}(\alpha)\right| = d^n$ (not necessarily distinct) this equality becomes

$$\begin{aligned}
\prod_{w \in \left\{R_{(-(n+1))}(\alpha)\right\}} w &= \left(\frac{-Q(0)}{P_d} \right)^{d^n} \prod_{w \in \left\{R_{(-n)}(\alpha)\right\}} w \\
&= \left(\frac{-Q(0)}{P_d} \right)^{d^n} \cdot (\alpha) \left(\frac{-Q(0)}{P_d} \right)^{\left(\frac{d^n - 1}{d-1}\right)} \\
&= \alpha \left(\frac{-Q(0)}{P_d} \right)^{\left(\frac{d^{n+1} - 1}{d-1}\right)},
\end{aligned}$$

as desired. □

Theorem 2.3.5. For a given fully symmetric self-similar structure on a finitely ramified fractal K, let V_n denote its sequence of approximating graphs and let P_n denote the probabilistic graph Laplacian of V_n. Arising naturally from the spectral decimation process, there is a rational function $R(z)$, which satisfies the conditions of Theorem 2.3.4, finite sets $A, B \subset \mathbb{R}$ such that for all $\alpha \in A$, $\beta \in B$, and integers $n, k \geq 0$, there exist functions α_n and β_n^k such that the number of spanning trees of V_n is given by

$$\tau(V_n) = \left| \frac{\left(\prod_{j=1}^{|V_n|} d_j\right)}{\left(\sum_{j=1}^{|V_n|} d_j\right)} \left(\prod_{\alpha\in A} \alpha^{\alpha_n}\right) \left[\prod_{\beta\in B} \left(\prod_{k=0}^{n} \left(\beta \left(\frac{-Q(0)}{P_d}\right)^{\frac{d^k-1}{d-1}}\right)^{\beta_n^k}\right)\right] \right|$$
$$= \left| \frac{\left(\prod_{j=1}^{|V_n|} d_j\right)}{\left(\sum_{j=1}^{|V_n|} d_j\right)} \left(\prod_{\alpha\in A} \alpha^{\alpha_n}\right) \left[\prod_{\beta\in B} \left(\beta^{\sum_{k=0}^{n} \beta_n^k} \left(\frac{-Q(0)}{P_d}\right)^{\sum_{k=0}^{n} \beta_n^k \left(\frac{d^k-1}{d-1}\right)}\right)\right] \right| \tag{2.4}$$

where d is the degree of $R(z)$, P_d is the leading coefficient of the numerator of $R(z)$, $|V_n|$ is the number of vertices of V_n and d_j is the degree of vertex j in V_n.

Proof of Theorem 2.3.5. From Kirchhoff's matrix-tree theorem for probabilistic graph Laplacians 2.3.1, we know that

$$\tau(V_n) = \frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j} \prod_{j=1}^{|V_n|-1} \lambda_j$$

where λ_j are the non-zero eigenvalues of P_n.

Existence and uniqueness of the rational function $R(z)$ is given Proposition (2.3.2). After carrying out the inductive calculations using Proposition (2.3.3) items (1)-(8), we get the sets A and B, and the functions α_n and β_n^k.

To see that the sets A and B are finite. Recall that the functions $R(z)$ and $\phi(z)$ from Proposition (2.3.3) are rational, thus $R(z)$, $\phi(z)$, and $R(z)\phi(z)$ have finitely many zeroes, poles, and removable singularities. Also, since the matrix D, from writing P_1 in block form to define the Schur Complement, is finite, $\sigma(D)$ is finite. Following items (1)-(8) of Proposition (2.3.3) these observations imply that A and B are finite sets.

From Proposition (2.3.3) we know that

$$\{\lambda_j\}_{j=1}^{|V_n|-1} = \bigcup_{\alpha\in A}\{\alpha\}\bigcup_{\beta\in B}\left[\bigcup_{k=0}^{n}\{R_{-k}(\beta) : \beta_n^k \neq 0\}\right]$$

where the multiplicities of $\alpha \in A$ are given by α_n and the multiplicities of $\{R_{-k}(\beta)\}$ are given by β_n^k. Letting $\lambda_{|V_n|} = 0$.

From items (1)-(8) of Proposition (2.3.3) it follows that $\forall z \in \{R_{-k}(\beta)\}$ the multiplicity of z depends only on n and k, thus

$$\prod_{j=1}^{|V_n|-1}\lambda_j = \left(\prod_{\alpha\in A}\alpha^{\alpha_n}\right)\left[\prod_{\beta\in B}\left(\prod_{k=0}^{n}\left(\prod_{z\in\{R_{-k}(\beta)\}} z^{\beta_n^k}\right)\right)\right]$$

From Lemma 4.9 in [44], $R(0) = 0$. From Corollary 1 in [33], it follows that, if we write $R(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are relatively prime polynomials, then $deg(P(z)) > deg(Q(z))$. Thus, the conditions of Theorem 2.3.4 are satisfied, and applying this theorem gives

$$= \left(\prod_{\alpha\in A}\alpha^{\alpha_n}\right)\left[\prod_{\beta\in B}\left(\prod_{k=0}^{n}\left(\beta\left(\frac{-Q(0)}{P_d}\right)^{\frac{d^k-1}{d-1}}\right)^{\beta_n^k}\right)\right]$$

$$= \left(\prod_{\alpha \in A} \alpha^{\alpha_n}\right)\left[\prod_{\beta \in B}\left(\beta^{\sum_{k=0}^{n}\beta_n^k}\left(\frac{-Q(0)}{P_d}\right)^{\sum_{k=0}^{n}\beta_n^k\left(\frac{d^k-1}{d-1}\right)}\right)\right]$$

Applying Kirchhoff's matrix-tree theorem for probabilistic graph Laplacians (Theorem 2.3.1), we verify the result.

□

2.4 Asymptotic Complexity

Lemma 2.4.1. For any two finite, connected graphs G_1, G_2, let $G_1 \vee_{x_1,x_2} G_2$ denote the graph formed by identifying the vertex $x_1 \in G_1$ with vertex $x_2 \in G_2$. Then $\forall x_1 \in G_1, x_2 \in G_2$

$$\tau(G_1 \vee_{x_1,x_2} G_2) = \tau(G_1) \cdot \tau(G_2) \tag{2.5}$$

Proof of Lemma 2.4.1. Any spanning tree of $G_1 \vee_{x_1,x_2} G_2$ when restricted to G_1 is a spanning tree of G_1, and similarly for G_2, so

$$\tau(G_1 \vee_{x_1,x_2} G_2) \leq \tau(G_1) \cdot \tau(G_2).$$

For any spanning trees T_1, T_2 of G_1 and G_2 respectively, $T_1 \vee_{x_1,x_2} T_2$ is a spanning tree of $G_1 \vee_{x_1,x_2} G_2$. This gives

$$\tau(G_1 \vee_{x_1,x_2} G_2) \geq \tau(G_1) \cdot \tau(G_2),$$

as desired. □

Dropping the assumption of full symmetry, we lose the spectral decimation process, but still have the following.

Theorem 2.4.2. For a given self-similar structure on a finitely ramified fractal K, let V_n denote its sequence of approximating graphs. Let m denote the number of 0-cells of the V_1 graph.

1. If V_1 is a tree, then $\tau(V_n) = 1 \ \forall n \geq 0$
2. If V_1 is not a tree, then $log(\tau(V_n)) \in \theta(|V_n|) = \theta(m^n)$

Proof of Theorem 2.4.2. If V_1 is a tree, then K is a fractal string. Hence $\forall n \geq 0$ V_n is a tree. If V_1 is not a tree, it is m copies of the V_0 graph with vertices identified appropriately. Similarly the V_n graph is m^n copies of the V_0 graph with vertices identified appropriately. Let $V_0 \vee_{x,x}^{m^n} V_0$ denote m^n copies of V_0 each identified to each other at some vertex $x \in V_0$, then clearly for $n \geq 0$

$$\tau(V_n) \geq \tau(V_0 \vee_{x,x}^{m^n} V_0). \tag{2.6}$$

Since V_1 is not a tree, $|V_0| > 2$, also the V_0 graph is the complete graph on $|V_0|$ vertices, so by Cayley's formula [60] $\tau(V_0) = |V_0|^{(|V_0|-2)}$.

Combining this with Proposition 2.4.1 we get that

$$\tau(V_0 \vee_{x,x}^{m^n} V_0) = |V_0|^{(|V_0|-2)\cdot m^n}$$

and

$$\tau(V_n) \geq |V_0|^{(|V_0|-2)\cdot m^n}.$$

So for $n \geq 0$,

$$log(\tau(V_n)) \geq m^n \cdot (|V_0| - 2)log(|V_0|) \sim |V_n| \tag{2.7}$$

Since $m^n \sim |V_n|$

Now, V_n can also be constructed by deletion of edges from the graph $K_{|V_n|}$. The deletion-contraction principle [60] says that for any connected graph G and any edge e in that graph

$$\tau(G) = \tau(G\backslash e) + \tau(G - e),$$

where $G\backslash e$ is the graph formed by contracting e in G and $G - e$ is the graph formed by deleting e from G.

This tells us that deleting edges from graphs decreases the number of spanning trees, thus

$$\tau(V_n) \leq \tau(K_{|V_n|}) = |V_n|^{(|V_n|-2)}.$$

Since $|V_n| \sim m^n$,

$$\tau(V_n) \lesssim m^{n(m^n-2)},$$

which implies $\forall \epsilon > 0$

$$\lim_{x\to\infty} \frac{log(\tau(V_n))}{m^{n(1+\epsilon)}} = 0. \tag{2.8}$$

Now, suppose that the sequence $\frac{log(\tau(V_n))}{m^n}$ is unbounded then $\forall M > 0$ $\exists n_0$ s.t. $\forall n \geq n_0$ $\frac{log(\tau(V_n))}{m^n} > M$, but then $\forall \epsilon > 0$ and $\forall n > \frac{n_0}{(1+\epsilon)}$, $\frac{log(\tau(V_n))}{m^{n(1+\epsilon)}} > M$ which contradicts equation 2.8. Thus, $\frac{log(\tau(V_n))}{m^n}$ is bounded and combining this with equation 2.7 implies $\log(\tau(V_n)) \in \theta(|V_n|)$, as desired.

□

Corollary 2.4.3. For a given self-similar structure on a finitely ramified fractal K, let V_n denote its sequence of approximating graphs. The following limits exist.

$$\limsup_{n\to\infty} \frac{log(\tau(V_n))}{|V_n|}, \tag{2.9}$$

$$\liminf_{n\to\infty} \frac{log(\tau(V_n))}{|V_n|}. \tag{2.10}$$

Proof. This follows immediately from Theorem 2.4.2. □

Theorem 2.4.4. There is no upper bound on the asymptotic complexity constant c_K for the class of finitely ramified fractals with self-similar structure.

Proof. From Corollary 3.6.2, the m-Tree Fractal, for $m \geq 3$, has an asymptotic complexity constant of

$$c_{K_m} = \frac{(m-2) \cdot log(m)}{(m-1)}, \tag{2.11}$$

which grows arbitrarily large as m tends to infinity. □

CHAPTER 3

APPLICATIONS: COUNTING SPANNING TREES FOR SPECIFIC FRACTALS

3.1 Sierpiński Gasket

The Sierpiński gasket has been extensively studied (in [53, 4, 37, 46, 7, 21, 26, 50, 54], among others.) It can be constructed as a p.c.f. fractal, in the sense of Kigami [37], in $\mathbb{R}^2$ using the contractions

$$f_1(x) = \frac{1}{2}(x - q_1) + q_1$$
$$f_2(x) = \frac{1}{2}(x - q_2) + q_2$$
$$f_3(x) = \frac{1}{2}(x - q_3) + q_3$$

where the points q_i are the vertices of an equilateral triangle.

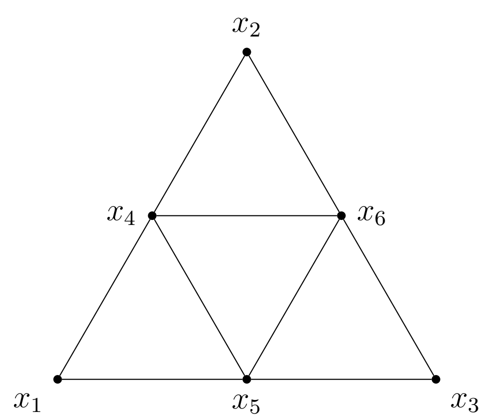

Figure 3.1: The V_1 network of the Sierpiński gasket.

In [16], the following theorem was proven. Here we give a new proof using the method described in Chapter 2.

Theorem 3.1.1. The number of spanning trees on the Sierpiński gasket at level n is given by

$$\tau(V_n) = 2^{f_n} \cdot 3^{g_n} \cdot 5^{h_n}, \qquad n \geq 0$$

where

$$\begin{aligned} f_n &= \frac{1}{2}\left(3^n - 1\right) \\ g_n &= \frac{1}{4}\left(3^{n+1} + 2n + 1\right) \\ h_n &= \frac{1}{4}\left(3^n - 2n - 1\right). \end{aligned}$$

Proof of Theorem 3.1.1. We apply Theorem 2.3.5.

It is well known that the V_n network of the Sierpiński gasket has

$$|V_n| = \frac{3^{n+1}+3}{2} \qquad n \geq 0$$

vertices, three of which have degree 2 and the remaining vertices have degree 4. So we compute

$$\prod_{i=1}^{|V_n|} d_i = 2^3 \cdot 4^{\frac{3^{n+1}+3}{2}-3} = 2^{3^{n+1}} \tag{3.1}$$

$$\sum_{i=1}^{|V_n|} d_i = 2 \cdot 3 + 4\left(\frac{3^{n+1}-3}{2}\right) = 2 \cdot 3^{n+1}. \tag{3.2}$$

Hence,

$$\frac{\prod_{i=1}^{|V_n|} d_i}{\sum_{i=1}^{|V_n|} d_i} = 2^{3^{n+1}-1} \cdot 3^{-(n+1)}. \tag{3.3}$$

In [3], they use a result from [4] to carry out spectral decimation for the Sierpiński gasket. In our language, they showed that

$$A = \left\{\frac{3}{2}\right\}$$
$$B = \left\{\frac{3}{4}, \frac{5}{4}\right\},$$

(I) $\alpha = \frac{3}{2}$, $\alpha_n = \frac{3^n+3}{2}$, $\quad n \geq 0$,

(II) $\beta = \frac{3}{4}$, $\quad n \geq 1$

$$\beta_n^k = \begin{cases} \frac{3^{n-k-1}+3}{2} & k = 0, \dots, n-1 \\ 0 & k = n, \end{cases}$$

(III) $\beta = \frac{5}{4}$, $\quad n \geq 2$

$$\beta_n^k = \begin{cases} \frac{3^{n-k-1}-1}{2} & k = 0, \dots, n-2 \\ 0 & k = n-1, n \end{cases}$$

and $R(z) = z(5-4z)$. So $d = 2$, $Q(0) = 1$ and $P_d = -4$.

We now use Equation 2.4 in Theorem 2.3.5 to calculate $\tau(V_n)$. We have

$$\prod_{\alpha \in A} \alpha^{\alpha_n} = \left(\frac{3}{2}\right)^{\dfrac{3^n+3}{2}} \tag{3.4}$$

$$\prod_{\beta \in B} \left(\beta^{\sum_{k=0}^{n} \beta_n^k} \cdot \left(\frac{1}{4}\right)^{\sum_{k=0}^{n} \beta_n^k \left(2^k - 1\right)} \right) =$$
$$= \left(\frac{3}{4}\right)^{\sum_{k=0}^{n-1} \left(\dfrac{3^{n-k-1}+3}{2}\right)} \times \left(\frac{1}{4}\right)^{\sum_{k=0}^{n-1} \left(\dfrac{3^{n-k-1}+3}{2}\right) \left(2^k - 1\right)} \tag{3.5}$$
$$\times \left(\frac{5}{4}\right)^{\sum_{k=0}^{n-2} \left(\dfrac{3^{n-k-1}-1}{2}\right)} \times \left(\frac{1}{4}\right)^{\sum_{k=0}^{n-2} \left(\dfrac{3^{n-k-1}-1}{2}\right) \left(2^k - 1\right)}$$

We sum the expressions in the exponents above.

$$\sum_{k=0}^{n-1}\left(\frac{3^{n-k-1}+3}{2}\right) = \frac{1}{4}\left(3^n+6n-1\right)$$
$$\sum_{k=0}^{n-1}\left(\frac{3^{n-k-1}+3}{2}\right)\left(2^k-1\right) = \frac{1}{4}\left(3^n+2^{n+2}-6n-5\right)$$
$$\sum_{k=0}^{n-2}\left(\frac{3^{n-k-1}-1}{2}\right) = \frac{1}{4}\left(3^n-2n-1\right)$$
$$\sum_{k=0}^{n-2}\left(\frac{3^{n-k-1}-1}{2}\right)\left(2^k-1\right) = \frac{1}{4}\left(3^n-2^{n+2}+2n+3\right).$$

All of these equations are valid for $n \geq 2$. Using equations 2.4, 3.3, 3.4,and 3.5, and simplifying we get:

$$\tau(V_n) = 2^{f_n}\cdot 3^{g_n}\cdot 5^{h_n} \qquad n \geq 2,$$

as desired.For $n = 1$, equation 3.3 still holds and the eigenvalues of the probabilistic graph Laplacian are $\{\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{4},\frac{3}{4},0\}$. So by Theorem 2.3.1, we get that $\tau(V_1) = 2\cdot 3^3$. The V_0 network is the complete graph on 3 vertices, thus $\tau(V_0) = 3$. Hence the theorem holds for all $n \geq 0$. □

As in [16], we immediately have the following Corollary.

Corollary 3.1.2. The asymptotic growth constant for the Sierpiński Gasket is

$$c = \frac{log(2)}{3}+\frac{log(3)}{2}+\frac{log(5)}{6} \tag{3.6}$$

Proof. Use Theorem 3.1.1 and recall that

$$|V_n| = \frac{3^{n+1}+3}{2} \qquad n \geq 0$$

□

3.2 A Non-p.c.f. Analog of the Sierpiński Gasket

As described in [4, 56, 8], this fractal is finitely ramified by not p.c.f. in the sense of Kigami. It can be constructed as a self-affine fractal in $\mathbb{R}^2$ using 6 affine contractions. One affine contraction has the fixed point $(0,0)$ and the matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

and the other five affine contractions can be obtained though combining this one with the symmetries of the equilateral triangle on vertices $(0,0)$, $(1,0)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Figure 3.2 shows the V_1 network for this fractal.

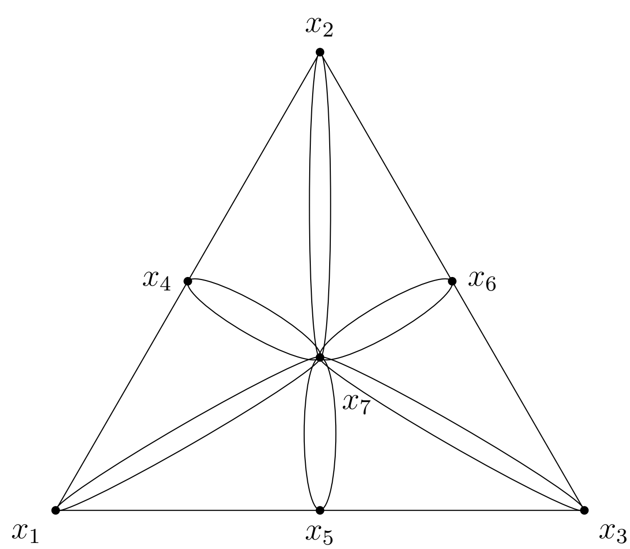

Figure 3.2: The V_1 network of the non-p.c.f. analog of the Sierpiński gasket.

Theorem 3.2.1. The number of spanning trees on the non-p.c.f. analog of the Sierpiński gasket at level n is given by

$$\tau(V_n) = 2^{f_n} \cdot 3^{g_n} \cdot 5^{h_n}, \quad n \geq 0$$

where

$$
\begin{aligned}
f_n &= \frac{2}{25}\left(11 \cdot 6^n - 30n - 11\right), \\
g_n &= \frac{1}{5}\left(2 \cdot 6^n + 3\right), \text{ and} \\
h_n &= \frac{1}{25}\left(4 \cdot 6^n + 30n - 4\right).
\end{aligned}
$$

Before the proof, we need a few results.

Theorem 3.2.2. The V_n network of the non-p.c.f. analog of the Sierpiński gasket, for $n \geq 0$, has

$$\frac{4 \cdot 6^n + 11}{5}$$

vertices. Among these vertices,

(i) 3 have degree 2^{n+1},

(ii) 6^{k-1} have degree $3 \cdot 2^{n-k+2}$ for $1 \leq k \leq n$, and

(iii) $3 \cdot 6^{k-1}$ have degree 2^{n-k+2} for $1 \leq k \leq n$.

Proof of Theorem 3.2.2. We first describe how the V_n network is constructed, then prove the Theorem.

For $n = 0$, V_0 is the complete graph on vertices $\{x_1, x_2, x_3\}$, one triangle (the V_0 network) and 3 corners of degree 2 $\{x_1, x_2, x_3\}$ are born at level 0.

For $n = 1$, from the triangle born on level 0, 6 triangles are born. For example one of these triangles is the complete graph on $\{x_2, x_4, x_7\}$. 3 corners of degree 4 are born, they are $\{x_4, x_5, x_6\}$ and one center is born $\{x_7\}$ of degree 12.

For $n \geq 2$, from each triangle born at level $n-1$, 6 triangles are born, 3 corners of degree 4 are born and 1 center of degree 12 is born. Each corner born at level $n-1$ gains 4 edges. Each center born at level $n-1$ gains 12 edges. Each corner born at level $n-2$ gains $2 \cdot 4$ edges. Each center born at level $n-2$ gains $2 \cdot 12$ edges. In general, for $1 \leq k \leq n-1$, each corner born at level $n-k$ gains $2^{k-1} \cdot 4$ edges, and each center born at level $n-k$ gains $2^{k-1} \cdot 12$ edges. The corners born at level 0 gain 2^n edges.

From this construction we see that, for $n \geq 0$ the V_n network has

$$3 + 4 \cdot \sum_{j=0}^{n-1} 6^j = \frac{4 \cdot 6^n + 11}{5}$$

vertices, as desired.

On the V_n network, for $n \geq 0$, the 3 corners born on level 0 have degree

$$2 + \sum_{j=1}^{n} 2^j = 2^{n+1},$$

which verifies item (i).

Following the construction, we see that on the V_n network, for $n \geq 1$, there are 6^{n-1} centers born at level n, each with degree 12. There are 6^{n-2} centers born at level $n-1$, each with degree $12 + 12$. In general, for $0 \leq k \leq n$, there are 6^{n-k-1} centers born at level n-k, each with degree

$$12 + 12 \cdot \sum_{j=0}^{k-1} 2^j = 3 \cdot 2^{k+2}.$$

After changing indices, item (ii) follows, noting that item (ii) is a vacuous statement for $n = 0$.

Similarly, for $0 \leq k \leq n$, in the V_n network, there are $3 \cdot 6^{n-k-1}$ corners born at level $n-k$. Each of which have degree

$$4 + 4 \cdot \sum_{j=0}^{k-1} 2^j = 2^{k+2}.$$

After changing indices, item (iii) follows, noting that item (iii) is a vacuous statement for $n = 0$. $\square$

Corollary 3.2.3. For the V_n network of the non-p.c.f. analog of the Sierpiński gasket, we have

$$\frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j} = 2^{\frac{1}{25}(44 \cdot 6^n + 30n + 6)} \cdot 3^{\frac{1}{5}(6^n - 5n - 6)}$$

for $n \geq 1$.

Proof of Corollary 3.2.3. From Theorem 3.2.2, we know that

$$\begin{aligned}
\prod_{j=1}^{|V_n|} d_j &= \left(2^{n+1}\right)^3 \cdot \prod_{k=1}^{n} \left(3 \cdot 2^{n-k+2}\right)^{6^{k-1}} \cdot \prod_{k=1}^{n} \left(2^{n-k+2}\right)^{3 \cdot 6^{k-1}} \\
&= 2^{3n+3} \cdot 3^{\sum_{k=1}^{n} 6^{k-1}} \cdot 2^{4 \cdot \sum_{k=1}^{n} (n-k+2) \cdot 6^{k-1}} \\
&= 2^{\frac{1}{25}(44 \cdot 6^n + 55n + 31)} \cdot 3^{\frac{1}{5} \cdot (6^n - 1)}.
\end{aligned}$$

It also follows from the previous proposition that

$$\begin{aligned}
\sum_{j=1}^{|V_n|} d_j &= 3 \cdot 2^{n+1} + \sum_{k=1}^{n} 6^{k-1} \left(3 \cdot 2^{n-k+2}\right) + 3 \cdot \sum_{k=1}^{n} 6^{k-1} \left(2^{n-k+2}\right) \\
&= 3 \cdot 2^{n+1} + \sum_{k=1}^{n} 6^{k} \cdot 2^{n-k+2} \\
&= 3 \cdot 2^{n+1} + 2^{n+2} \sum_{k=1}^{n} 3^{k} \\
&= 3 \cdot 2^{n+1} \left(1 + 2 \sum_{k=1}^{n} 3^{k-1}\right) \\
&= 2^{n+1} \cdot 3^{n+1}.
\end{aligned}$$

Combining these calculations, the Corollary follows. □

We are now ready for the proof of the main theorem in this section.

Proof of Theorem 3.2.1. We apply Theorem 2.3.5. In [4], they use a result from [3] to carry out spectral decimation for the non-p.c.f. analog of the Sierpiński gasket. In our language, they showed that

$$A = \left\{ \frac{3}{2} \right\},$$
$$B = \left\{ \frac{3}{4}, \frac{5}{4}, \frac{1}{2}, 1 \right\}.$$

Rephrasing their results in our language, for $n \geq 2$ the following hold:

(I) $\alpha = \frac{3}{2}$, $\alpha_n = 6^{n-1} + 1$,

(II) $\beta = \frac{3}{4}$,

$$\beta_n^k = \begin{cases} 6^{n-k-2} + 1 & k = 0, \dots, n-2 \\ 2 & k = n-1 \\ 0 & k = n, \end{cases}$$

(III) $\beta = \frac{5}{4}$,

$$\beta_n^k = \begin{cases} 6^{n-k-2} + 1 & k = 0, \dots, n-2 \\ 2 & k = n-1 \\ 0 & k = n, \end{cases}$$

(IV) $\beta = \frac{1}{2}$,

$$\beta_n^k = \begin{cases} \dfrac{11 \cdot 6^{n-k-2} - 6}{5} & k = 0, \dots, n-2 \\ 0 & k = n-1, n, \end{cases}$$

(V) $\beta = 1$,

$$\beta_n^k = \begin{cases} \dfrac{6^{n-k} - 6}{5} & k = 0, \dots, n-2 \\ 0 & k = n-1, n, \end{cases}$$

and

$$R(z) = \frac{-24z(z-1)(2z-3)}{14z-15}.$$

So $d = 3$, $Q(0) = -15$ and $P_d = 48$.

We now use Equation 2.4 in Theorem 2.3.5 to calculate $\tau(V_n)$. We have from (I),

$$\prod_{\alpha \in A} \alpha^{\alpha_n} = \left(\frac{3}{2}\right)^{6^{n-1}+1}. \tag{3.7}$$

From (II),(III),(V), and (V), we have that

$$\prod_{\beta \in B} \left(\beta^{\sum_{k=0}^{n} \beta_n^k} \cdot \left(\frac{15}{48}\right)^{\sum_{k=0}^{n} \beta_n^k \left(\frac{d^k-1}{d-1}\right)} \right) =$$

$$\begin{aligned}
&= \left(\frac{3}{4}\right)^{\left[\sum_{k=0}^{n-2} \left(6^{n-k-2}+1\right)\right]+2} \times \left(\frac{15}{48}\right)^{\left[\sum_{k=0}^{n-2} \left(6^{n-k-2}+1\right)\left(\frac{3^k-1}{2}\right)\right]+2\cdot\frac{3^{n-1}-1}{2}} \\
&\times \left(\frac{5}{4}\right)^{\left[\sum_{k=0}^{n-2} \left(6^{n-k-2}+1\right)\right]+2} \times \left(\frac{15}{48}\right)^{\left[\sum_{k=0}^{n-2} \left(6^{n-k-2}+1\right)\left(\frac{3^k-1}{2}\right)\right]+2\cdot\frac{3^{n-1}-1}{2}} \\
&\times \left(\frac{1}{2}\right)^{\sum_{k=0}^{n-2} \frac{11\cdot 6^{n-k-2}-6}{5}} \times \left(\frac{15}{48}\right)^{\sum_{k=0}^{n-2} \left(\frac{11\cdot 6^{n-k-2}-6}{5}\right)\left(\frac{3^k-1}{2}\right)} \\
&\times \left(1\right)^{\sum_{k=0}^{n-2} \frac{6^{n-k}-6}{5}} \times \left(\frac{15}{48}\right)^{\sum_{k=0}^{n-2} \left(\frac{6^{n-k}-6}{5}\right)\left(\frac{3^k-1}{2}\right)}
\end{aligned} \tag{3.8}$$

We sum the expression in the exponents above.

$$\left[\sum_{k=0}^{n-2}\left(6^{n-k-2}+1\right)\right]+2=\frac{1}{5}\left(6^{n-1}+5n+4\right)$$

$$\left[\sum_{k=0}^{n-2}\left(6^{n-k-2}+1\right)\left(\frac{3^k-1}{2}\right)\right]+\left(3^{n-1}-1\right)=\frac{1}{60}\left(4\cdot 6^{n-1}+65\cdot 3^{n-1}-30n-39\right)$$

$$\sum_{k=0}^{n-2}\frac{11\cdot 6^{n-k-2}-6}{5}=\frac{1}{25}\left(11\cdot 6^{n-1}-30n+19\right)$$

$$\sum_{k=0}^{n-2}\left(\frac{11\cdot 6^{n-k-2}-6}{5}\right)\left(\frac{3^k-1}{2}\right)=\frac{1}{25}\left(22\cdot 6^{n-2}-50\cdot 3^{n-2}+15n-2\right)$$

$$\sum_{k=0}^{n-2}\left(\frac{6^{n-k}-6}{5}\right)\left(\frac{3^k-1}{2}\right)=\frac{1}{50}\left(4\cdot 6^{n}-25\cdot 3^{n}+30n+21\right)$$

All of these equations are valid for $n \geq 2$ and combining with Corollary 3.2.3, we see that

$$\tau(V_n)=2^{f_n}\cdot 3^{g_n}\cdot 5^{h_n},\quad n\geq 2$$

where

$$\begin{aligned} f_n &= \frac{2}{25}\left(11\cdot 6^n-30n-11\right), \\ g_n &= \frac{1}{5}\left(2\cdot 6^n+3\right), \text{ and} \\ h_n &= \frac{1}{25}\left(4\cdot 6^n+30n-4\right). \end{aligned}$$

For $n = 0$, since the V_0 graph is the complete graph on three vertices, $\tau(V_0) = 3$ by Cayley's Formula, as desired. For $n = 1$, from [4] the eigenvalues of P_1 are $\{\frac{5}{4}, \frac{5}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{4}, \frac{3}{4}, 0\}$ and using Corollary 3.2.3 for $n = 1$, we apply Theorem 2.3.1 to see that $\tau(V_1) = 2^2 \cdot 3^3 \cdot 5^2$, as desired. □

Corollary 3.2.4. The asymptotic growth constant for the non-p.c.f. analog of the Sierpiński Gasket is

$$c=\frac{11\cdot log(2)}{10}+\frac{log(3)}{2}+\frac{log(5)}{5} \tag{3.9}$$

Proof. Use Theorem 3.2.1 and recall that

$$|V_n| = \frac{4 \cdot 6^n + 11}{5}$$

□

3.3 Diamond Fractal

The diamond self-similar hierarchical lattice appeared as an example in several physics works, including [28], [29], and [27]. In[3] the authors modify the standard results for the unit interval $[0, 1]$ to develop the spectral decimation method for this fractal, hence Theorem 2.3.5 still applies. Figure 3.3 shows the V_1 and V_2 networks for this.

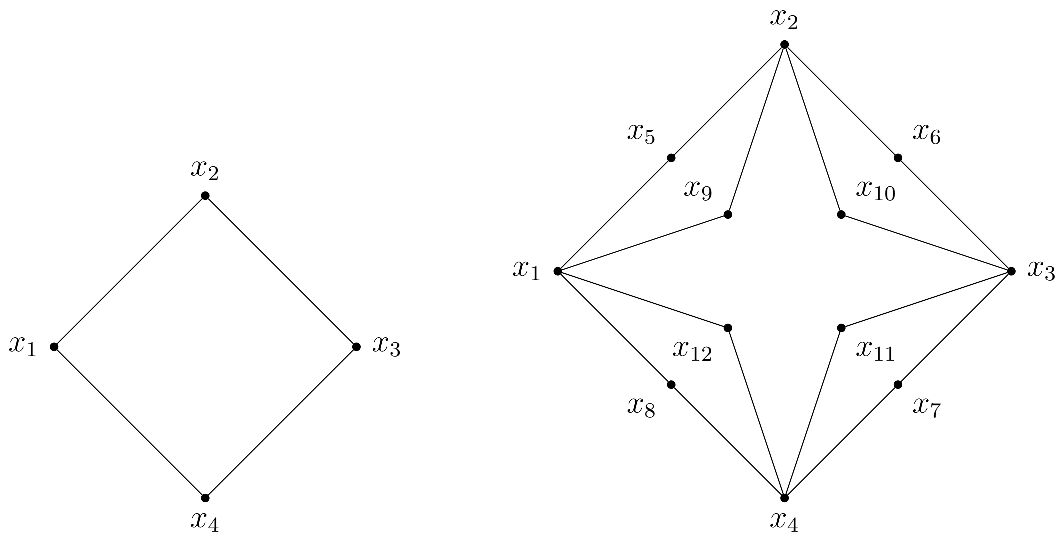

Figure 3.3: The V_1 and V_2 network of the Diamond fractal.

Theorem 3.3.1. The number of spanning trees on the Diamond fractal at level n is given by

$$\tau(V_n) = 2^{\frac{2}{3}(4^n - 1)} \qquad n \geq 1.$$

Before we begin the proof, we need a few results.

Theorem 3.3.2. The V_n network of the Diamond fractal, for $n \geq 1$, has

$$\frac{(4 + 2 \cdot 4^n)}{3}$$

vertices. Among these vertices,

(i) $2 \cdot 4^{n-k}$ have degree 2^k for $1 \leq k \leq n-1$

(ii) 4 have degree 2^n.

Remark 3.3.3. In [3], the number of vertices of V_n is incorrect as stated in Theorem 7.1(ii). We correct this here and provide a proof.

Proof of Theorem 3.3.2. We first describe how the V_n network is constructed, then prove the Proposition. When $n = 1$, V_1 has four vertices of degree 2 and 1 diamond, this diamond is the graph of V_1. We say these vertices and diamond are born at level 1.

When $n = 2$, from the diamond born on level 1, 4 diamonds are born. We say these diamonds are born on level 2. For each of the diamonds born on level 2, 2 vertices of degree 2 are born. We say these vertices are born on level 2. Using the notation $G =< V, E >$ where G is the graph, V is the graph's vertex set and E is the graph's edge set. An example diamond born at level 2 is $< V, E >$, where

$$V = \{x_1, x_5, x_2, x_9\}$$

$$E = \{x_1x_5, x_5x_2, x_2x_9, x_9x_1\}$$

which gives birth to x_5 and x_9. Every vertex born on level 1 gains 2 more edges.

For $n \geq 2$, from each diamond born on level $n-1$, 4 diamonds are born at level n. For each of the diamonds born on level n, 2 vertices of degree 2 are born at level n. Every vertex born on level $n-1$, gains 2 more edges. Every vertex born on level $n-2$, gains 2^2 more edges. In general, every vertex born on level $n-k$, gains 2^k more edges for $1 \leq k \leq n-1$.

From this construction, we see that at level n, for $n \geq 1$, there are 4^{k-1} diamonds born at level k, $1 \leq k \leq n$, $2 \cdot 4^{k-1}$ vertices born at level k, $2 \leq k \leq n$ and 4 vertices born at level 1. Thus, the V_n network has

$$4 + \sum_{k=2}^{n} 2 \cdot 4^{k-1} = \frac{(4 + 2 \cdot 4^n)}{3} \text{vertices, as desired.}$$

In the V_n network, the 4 vertices born at level 1 have degree

$$2 + \sum_{j=1}^{n-1} 2^j = 2^n,$$

which verifies item (ii) of the Proposition.

In the V_n network, the $2 \cdot 4^{k-1}$ vertices born on level k, $2 \leq k \leq n$, have degree

$$2 + \sum_{j=1}^{n-k} 2^j = 2^{n-k+1}.$$

changing indices, this verifies item (i) of the Proposition. □

Corollary 3.3.4. For the V_n network of the Diamond fractal, we have

$$\frac{\prod_{i=1}^{|V_n|} d_i}{\sum_{i=1}^{|V_n|} d_i} = 2^{\frac{1}{9}\left(2 \cdot 4^{n+1} - 6n - 17\right)}. \tag{3.10}$$

Proof of Corollary 3.3.4. From Theorem 3.3.2, we know that

$$\begin{aligned}
\prod_{i=1}^{|V_n|} d_i &= (2^n)^4 \cdot \prod_{k=1}^{n-1} \left(2^k\right)^{2 \cdot 4^{n-k}} \\
&= 2^{4n} \cdot 2^{\sum_{k=1}^{n-1} 2 \cdot k \cdot 4^{n-k}} \\
&= 2^{\frac{1}{9}\left(2 \cdot 4^{n+1} + 12n - 8\right)}
\end{aligned}$$

It also follows from the previous proposition that

$$\begin{aligned}
\sum_{i=1}^{|V_n|} d_i &= \left(\sum_{k=1}^{n-1} 2 \cdot 2^k \cdot 4^{n-k}\right) + 4 \cdot 2^n \\
&= 2^{2n+1}.
\end{aligned}$$

Combining these calculations, the corollary follows. □

We now return to a proof the the main theorem of this section.

Proof of Theorem 3.3.1. We apply Theorem 2.3.5. In [3], they carry out spectral decimation for the Diamond fractal. In our language, they showed that

$$A = \{2\}$$
$$B = \{1\}.$$

For $n \geq 1$, the following hold:

(I) $\alpha = 2$, $\alpha_n = 1$

(II) $\beta = 1$,

$$\beta_n^k = \begin{cases} \frac{4^{n-k}+2}{3} & k = 0, \ldots, n-1 \\ 0 & k = n, \end{cases}$$

and

$$R(z) = 2z(2-z).$$

So $d = 2$, $Q(0) = 1$, and $P_d = -2$. We now use Equation 2.4 in Theorem 2.3.5 to calculate $\tau(V_n)$.

$$\prod_{\alpha \in A} \alpha^{\alpha_n} = 2^1 \tag{3.11}$$

$$\prod_{\beta \in B} \left(\beta^{\sum_{k=0}^{n} \beta_n^k} \cdot \left(\frac{1}{2}\right)^{\sum_{k=0}^{n} \beta_n^k \left(2^k - 1\right)} \right) = \\ = 1^{\sum_{k=0}^{n-1} \left(\frac{4^{n-k}+2}{3}\right)} \times \left(\frac{1}{2}\right)^{\sum_{k=0}^{n-1} \left(\frac{4^{n-k}+2}{3}\right) \left(2^k - 1\right)} \tag{3.12}$$

We sum the relevant expression in the exponents above:

$$\sum_{k=0}^{n-1} \left(\frac{4^{n-k}+2}{3}\right) \left(2^k - 1\right) = \frac{1}{9} \left(2 \cdot 4^n - 6n - 2\right).$$

Combining this with Corollary 3.3.4, we have that

$$\tau(V_n) = 2^{\frac{2}{3}(4^n-1)} \qquad n \geq 1$$

as desired. □

Corollary 3.3.5. The asymptotic growth constant for the Diamond fractal is

$$c = log(2) \tag{3.13}$$

Proof. Use Theorem 3.3.1 and recall that

$$|V_n| = \frac{(4 + 2 \cdot 4^n)}{3}$$

□

3.4 Hexagasket

The hexagasket, is also known as the hexakun, a polygasket, a 6-gasket, or a $(2,2,2)$-gasket, see [4, 37, 1, 12, 53, 56, 61, 62]. The V_1 network of the hexagasket is shown in the figure below.

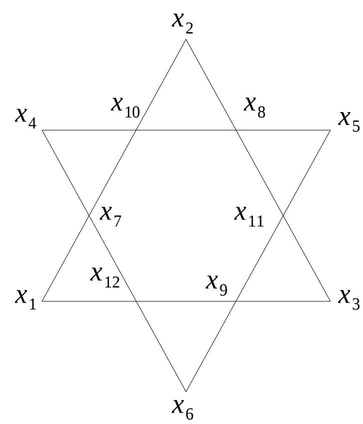

Figure 3.4: The V_1 network of the Hexagasket.

Theorem 3.4.1. The number of spanning trees on the Hexagasket at level n is given by

$$\tau(V_n) = 2^{f_n} \cdot 3^{g_n} \cdot 7^{h_n} \qquad n \geq 0.$$

where

$$\begin{aligned} f_n &= \frac{1}{225}\big(27 \cdot 6^{n+1} - 100 \cdot 4^n - 60n - 62\big) \\ g_n &= \frac{1}{25}\big(4 \cdot 6^{n+1} + 5n + 1\big) \\ h_n &= \frac{1}{25}\left(6^n - 5n - 1\right). \end{aligned}$$

Proof of Theorem 3.4.1. We apply Theorem 2.3.5. From [4] it is known that

$$|V_n| = \frac{(6+9\cdot 6^n)}{5} \qquad n \geq 0,$$

of these vertices

$$\frac{6(6^n-1)}{5} \quad \text{have degree } 4,$$

and the remaining

$$\frac{(12+3\cdot 6^n)}{5} \quad \text{have degree } 2.$$

So we compute

$$\frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j} = 2^{(3\cdot 6^n - n - 1)} \cdot 3^{-(n+1)} \tag{3.14}$$

for $n \geq 0$.

In [4], they use a result from [3] to carry out spectral decimation for the Hexagasket. We note that in [4] Theorem 6.1 (v) and (vi), the bounds on k should be $0 \leq k \leq n-1$ and in (vii) the bounds should be $0 \leq k \leq n-2$. This can be verified using Table 2 in the same paper. In our language they showed that

$$A = \left\{\frac{3}{2}\right\},$$
$$B = \left\{1, \frac{1}{4}, \frac{3}{4}, \frac{3+\sqrt{2}}{4}, \frac{3-\sqrt{2}}{4}\right\},$$

and for $n \geq 2$ the following hold:

(I) $\alpha = \frac{3}{2}$, $\quad \alpha_n = \frac{(6+4\cdot 6^n)}{5}$,

(II) $\beta = 1$,

$$\beta_n^k = \begin{cases} 1 & k = 0, \ldots, n-1 \\ 0 & k = n, \end{cases}$$

(III) $\beta = \frac{1}{4}, \frac{3}{4}$,

$$\beta_n^k = \begin{cases} \frac{(6+4\cdot 6^{n-k-1})}{5} & k = 0, \ldots, n-1 \\ 0 & k = n, \end{cases}$$

(IV) $\beta = \frac{3+\sqrt{2}}{4}, \frac{3-\sqrt{2}}{4}$,

$$\beta_n^k = \begin{cases} \frac{(6^{n-k-1}-1)}{5} & k = 0, \ldots, n-2 \\ 0 & k = n-1, n, \end{cases}$$

$$R(z) = \frac{2z(z-1)(7-24z+16z^2)}{(2z-1)}.$$

So $d = 4$, $Q(0) = -1$ and $P_d = 32$.

We now use equation 2.4 in Theorem 2.3.5 to calculate $\tau(V_n)$. The relevant sums are

$$\sum_{k=0}^{n-1} \frac{(4^k-1)}{3} = \frac{(4^n-3n-1)}{9} \tag{3.15}$$

$$\sum_{k=0}^{n-1} \frac{(6+4\cdot 6^{n-k-1})}{5} = \frac{2\cdot(2\cdot 6^n+15n-2)}{25} \tag{3.16}$$

$$\sum_{k=0}^{n-1} \frac{(6+4\cdot 6^{n-k-1})}{5}\frac{(4^k-1)}{3} = \frac{(6^{n+1}-30n-6)}{75} \tag{3.17}$$

$$\sum_{k=0}^{n-2} \frac{(6^{n-k-1}-1)}{5} = \frac{(6^n-5n-1)}{25} \tag{3.18}$$

$$\sum_{k=0}^{n-2} \frac{(6^{n-k-1}-1)}{5}\frac{(4^k-1)}{3} = \frac{(9\cdot 6^n-25\cdot 4^n+30n+16)}{450} \tag{3.19}$$

Combining these using equations 2.4 and 3.14, after simplifying we get

$$\tau(V_n) = 2^{f_n}\cdot 3^{g_n}\cdot 7^{h_n} \qquad n \geq 2.$$

Where $f_n, g_n,$ and h_n are as claimed.

For n=1, equation 3.14 still holds and from [4] we know the eigenvalues of the probabilistic graph Laplacian on V_1 are $\{1, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 0\}$. So by Theorem 2.3.1, we get that $\tau(V_1) = 2^2 \cdot 3^6$, thus the theorem holds for $n = 1$. The V_0 network is the complete graph on 3 vertices, thus $\tau(V_0) = 3$. Hence the theorem holds for all $n \geq 0$. □

Corollary 3.4.2. The asymptotic growth constant for the Hexagasket is

$$c = \frac{2 \cdot log(2)}{5} + \frac{8 \cdot log(3)}{15} + \frac{log(7)}{45} \tag{3.20}$$

Proof. Use Theorem 3.4.1 and recall that

$$|V_n| = \frac{(6 + 9 \cdot 6^n)}{5}$$

□

3.5 Level-3 Sierpiński Gasket

The Level-3 Sierpiński Gasket can be constructed as a p.c.f. fractal, in the sense of Kigami [37], in $\mathbb{R}^2$ using the contractions $f_1, f_2, ..., f_6$, where each f_i is the mapping from the equilateral triangle $\{x_1, x_2, x_3\}$ to the six smaller triangles in the same orientation. This fractal has been studied in [23, 3, 5, 31, 53]. The figure below depicts the V_1 network of the Level-3 Sierpiński Gasket.

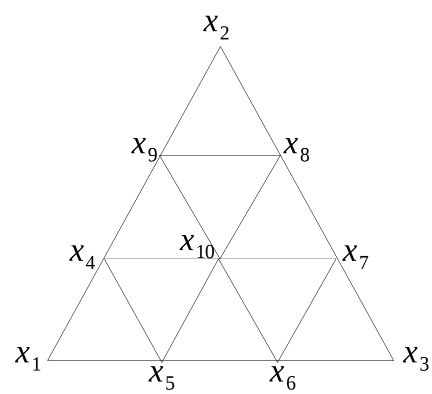

Figure 3.5: The V_1 network of the level-3 Sierpiński gasket.

Theorem 3.5.1. The number of spanning trees on the Level-3 Sierpiński Gasket at level n is given by

$$\tau(V_n) = 2^{f_n} \cdot 3^{g_n} \cdot 5^{h_n} \cdot 7^{i_n} \qquad n \geq 0.$$

where

$$f_n = \frac{2}{5}\left(6^n - 1\right)$$
$$g_n = \frac{1}{25}\left(13 \cdot 6^n - 15n + 12\right)$$
$$h_n = \frac{1}{25}\left(3 \cdot 6^n - 15n - 3\right).$$
$$i_n = \frac{1}{25}\left(7 \cdot 6^n + 15n - 7\right).$$

This theorem was originally proven in [16]. Here we give a new proof using the method described in Chapter 2.

Proof of Theorem 3.5.1. We apply Theorem 2.3.5.From [3] it is known that

$$|V_n| = 3 + \frac{7(6^n - 1)}{5} \qquad n \geq 0,$$

and it is easy to see that of these vertices

$$3 \quad \text{have degree } 2,$$

$$\frac{(6^n - 1)}{5} \quad \text{have degree } 6,$$

and

$$\frac{6(6^n - 1)}{5} \quad \text{have degree } 4.$$

So we compute

$$\frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j} = 2^{\frac{(13\cdot 6^n - 5n - 3)}{5}} \cdot 3^{\frac{(6^n - 5n - 6)}{5}} \tag{3.21}$$

for $n \geq 0$.

In [3] spectral decimation for this fractal is carried out. In our language they showed that

$$A = \left\{ \frac{3}{2} \right\},$$
$$B = \left\{ 1, \frac{3}{4}, \frac{5}{4}, \frac{3+\sqrt{2}}{4}, \frac{3-\sqrt{2}}{4}, \frac{3+\sqrt{5}}{4}, \frac{3-\sqrt{5}}{4} \right\},$$

and for $n \geq 2$ the following hold:

(I) $\alpha = \frac{3}{2}$, $\quad \alpha_n = \frac{(8+2\cdot 6^n)}{5}$,

(II) $\beta = 1$,

$$\beta_n^k = \begin{cases} 1 & k = 0, 1, 2 \\ 0 & k = 3, \ldots, n, \end{cases}$$

(III) $\beta = \frac{5}{4}, \frac{3}{4}$,

$$\beta_n^k = \begin{cases} \frac{3(6^{n-k-1}-1)}{5} & k = 0, \ldots, n-2 \\ 0 & k = n-1, n, \end{cases}$$

(IV) $\beta = \frac{3+\sqrt{2}}{4}, \frac{3-\sqrt{2}}{4}$,

$$\beta_n^k = \begin{cases} \frac{(2\cdot 6^{n-k-1}+8)}{5} & k = 0, \ldots, n-1 \\ 0 & k = n, \end{cases}$$

(V) $\beta = \frac{3+\sqrt{5}}{4}, \frac{3-\sqrt{5}}{4}$,

$$\beta_n^k = 0$$

$$R(z) = \frac{6z(z-1)(4z-5)(4z-3)}{(6z-7)}.$$

So $d = 4$, $Q(0) = -7$ and $P_d = 2^5 \cdot 3$.

We now use equation 2.4 in Theorem 2.3.5 to calculate $\tau(V_n)$. The relevant sums are

$$\sum_{k=0}^{n-1} \frac{(8+2\cdot 6^{n-k-1})}{5} = \frac{2\cdot(6^n+20n-1)}{25} \tag{3.22}$$

$$\sum_{k=0}^{n-1} \frac{(8+2\cdot 6^{n-k-1})}{5}\frac{(4^k-1)}{3} = \frac{(9\cdot 6^n+25\cdot 4^n-120n-34)}{225} \tag{3.23}$$

$$\sum_{k=0}^{n-2} \frac{3(6^{n-k-1}-1)}{5} = \frac{3(6^n-5n-1)}{25} \tag{3.24}$$

$$\sum_{k=0}^{n-2} \frac{3(6^{n-k-1}-1)}{5}\frac{(4^k-1)}{3} = \frac{(9\cdot 6^n-25\cdot 4^n+30n+16)}{150} \tag{3.25}$$

Combining these using equations 2.4 and 3.21, after simplifying we get

$$\tau(V_n) = 2^{f_n}\cdot 3^{g_n}\cdot 5^{h_n}7^{i_n} \qquad n\geq 2.$$

Where f_n, g_n, h_n, and i_n are as claimed.

For n=1, equation 3.21 still holds and from [3] we know the eigenvalues of the probabilistic graph Laplacian on V_1 are $\{1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3+\sqrt{2}}{4}, \frac{3+\sqrt{2}}{4}, \frac{3-\sqrt{2}}{4}, \frac{3-\sqrt{2}}{4}, 0\}$. So by Theorem 2.3.1, we get that $\tau(V_1) = 2^2\cdot 3^3\cdot 7^2$, thus the theorem holds for $n=1$. The V_0 network is the complete graph on 3 vertices, thus $\tau(V_0)=3$. Hence the theorem holds for all $n\geq 0$.

□

As in [16], we immediately have the following Corollary.

Corollary 3.5.2. The asymptotic growth constant for the Level-3Sierpiński Gasket is

$$c = \frac{2\cdot log(2)}{7} + \frac{13\cdot log(3)}{35} + \frac{3\cdot log(5)}{35} + \frac{log(7)}{5} \tag{3.26}$$

Proof. Use Theorem 3.5.1 and recall that

$$|V_n| = 3+\frac{7(6^n-1)}{5}$$

□

3.6 m-Tree Fractal, $m \geq 3$

The family of fractal trees indexed by the number of branches they possess provide a nice class of examples. In [24], Ford and Steinhurst carry out spectral decimation on them to describe the spectrum of the Laplacian on these trees. These examples show that even though each m-Tree Fractal in the limit is topologically a tree, the number of spanning trees on the approximating graphs grows arbitrarily large. Also, in Theorem 2.4.2 it is shown that for any given self-similar structure on a finitely ramified fractal the asymptotic complexity contant exist. The m-Tree Fractals show that there can be no uniform upper bound on the asymptotic complexity constant from Theorem 2.4.2.

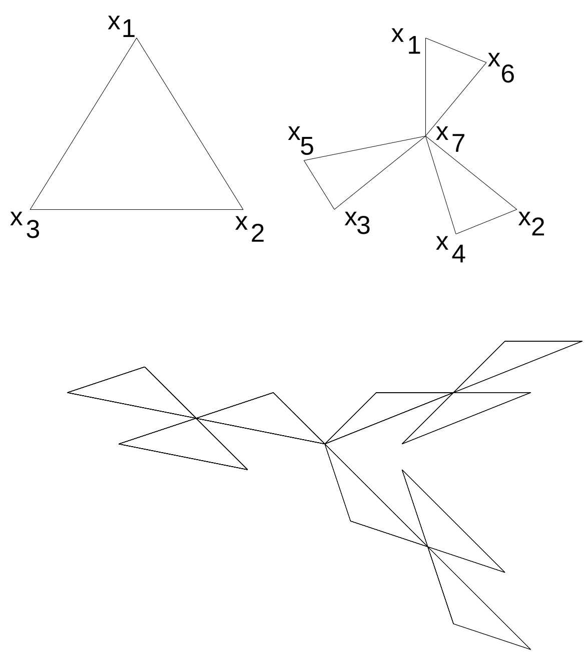

Figure 3.6: The $V_{3,0}$, $V_{3,1}$ and $V_{3,2}$ network of the 3-Tree Fractal

The m-Tree Fractal, K_m, is a fully symmetric finitely ramified self-similar structure with m defining contraction mappings. As in [24], the zero-level graph approximation, $V_{m,0}$, consists of a complete graph on m vertices. The iterated function system that generates the fractal scales, duplicates, and translates the simplex to m simplices sharing a common point at the epicenter of the previous simplex and with each vertex from $V_{m,0}$ as a vertex of one of the new simplices, this is the graph of $V_{m,1}$. This process is iterated and the countable set of vertices is completed in the effective resistance metric to form a tree with m branches. Let $V_{m,n}$ denote the n-th level approximating graph of K_m

Theorem 3.6.1. The number of spanning trees on the m-Tree Fractal, $m \geq 3$, at level n is given by

$$\tau(V_{m,n}) = m^{(m-2)\cdot m^n} \qquad n \geq 0.$$

While one could prove this in a similar manner to the previous examples using Theorem 2.3.5, and the spectral decimation carried out in [24], it is much easier to use Cayley's formula, [60], and Proposition 2.4.1.

Proof. From the construction of $V_{m,n}$, it is easy to see that $V_{m,n}$ is formed by m^n copies of $V_{m,0}$ (the complete graph on m vertices) wedged together in manner of Proposition 2.4.1. By Cayley's formula, [60], $\tau(V_{m,0}) = m^{m-2}$ so by Proposition 2.4.1, we have that

$$\tau(V_{m,n}) = m^{(m-2)\cdot m^n} \qquad n \geq 0,$$

as desired.

□

Corollary 3.6.2. The asymptotic growth constant for the m-Tree Fractal, K_m is

$$c_{K_m} = \frac{(m-2)\cdot log(m)}{(m-1)} \tag{3.27}$$

Proof. Use Theorem 3.6.1 and from Propositon 5.1 of [24],

$$|V_{m,n}| = 1 + (m-1) \cdot m^n \quad n \geq 0,$$

taking limits we are done. $\square$

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